AT-1 Time-Scale Algorithm for Optimization of Time and Frequency Stability from an Ensemble of Clocks

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Jim Barnes wrote the first atomic-clock time scale algorithm using three highly stable quartz-crystal-oscillator clocks calibrated with the Primary Frequency Standard. It was called NBS-A. He had solved the problem of dealing with flicker-noise in those clocks. The NBS PFS gave him away to calibrate the frequency offset and frequency drift in those clocks. The techniques he used are described in his Ph.D. thesis, which was published in February 1966 in a special issue of the IEEE entitled FREQUENCY STABILITY. My thesis was published there as well.

I quote some from page 283 and 284 from my book, *It's About Time.*

Jim had become Section Chief, and in 1968, I was responsible for using the ensemble of atomic clocks in a proper way for keeping time at NBS with the help of colleagues and inspiration I felt from above. I wrote what is called the AT1 timescale algorithm for generating official time for the USA; with many significant improvements made by my colleagues, this software clock is still being used today and generates official time, UTC(NIST). The $\sigma_y(\tau)$ versus τ diagram is very different for cesium-beam atomic clocks than for quartz-crystal-oscillator clocks that Jim had used. The AT1 algorithm was designed for the optimal use of atomic clocks.

The theory of the AT1 algorithm has some interesting features:

- The software clock has better performance than the best clock in the ensemble.
- The best clock cannot take over and is optimized.
- The worst clock is optimally used as well and enhances the output.
- The algorithm is robust and rejects clocks with malperformance.
- It optimizes both the short-term and long-term stability performance of each contributing clock by giving proper weights to each clock in the ensemble.
- The algorithm dynamically tracks the performance of each clock at each measurement cycle, which can change over time--especially when a cesium-beam atomic clock runs out of cesium and dies.
- This is like artificial intelligence, as it optimally updates the weights of all the clocks in the ensemble, as it looks at all the clocks at each measurement cycle.
- While AT-1 provides optimum ensemble frequency-stability performance, it needs a Primary Cesium Frequency Standard to calibrate its rate.
- Knowing this rate allows the generation of UTC(NBS/NIST) to be synchronized with the international official time scale, UTC.

Clocks are like people. All are different. One can think of clock weighting for optimum performance in the following way. One moves forward in life the best by giving the greatest weight to people's words who have the greatest wisdom and integrity. This idea is why trusting in the Lord and believing his words, who has perfect integrity and all wisdom, are fundamental to gaining light and truth. Similarly, clocks get weighted according to their stability--their predictability in time keeping--to give optimum performance for the ensemble time.

The following seven equations are how AT-1 ensemble times are computed. We take ensemble time to be the correct time, x_e. By definition, it has no time error. The hat over a variable denotes a best estimate value. The derivative of the time error of a clock is equal to the fractional frequency error for

that clock: y(t) = dx/dt. Or one can say that x(t) is the integral of y(t). The definition of y(t) = (v(t) − v$_e$)/v$_e$. So y$_i$(t) is the fractional frequency offset (rate) for the ith clock at time t with respect to the ensemble.

Equation 1. The hat over the x denotes the optimum prediction offset time of the ith clock at the current measurement cycle, (t$_o$ + τ), where tau, τ, is the interval between measurements. The hat over the y denotes the optimum estimate of its rate derived from the previous measurement cycle. The last term on the right of Equation 1 is x$_i$(t$_o$) and is the best estimate of the offset time for the ith clock as derived from the previous measurement cycle. In 1968, tau was one day, except for weekends. This was because we could only measure the time-differences between the clocks to a precision of 0.1 nanoseconds with the current counters then available. This made the measurement noise, 10^{-15}, less than the clock noise for tau equal one day.

1) $$\hat{x}_i\,(t_o + \tau) = \hat{y}_i\,(t_o) \cdot \tau + x_i(t_o)$$

Equation 2 uses the computation from equation 1 to calculate the optimum estimate of the offset time of the jth clock at the current measurement cycle. This is done for all the clocks in the ensemble, j = 1 to N. The weights always add up to 1 and are derived in equation 6 and are updated each measurement cycle. The weights are filtered through a digital exponential filter with a one-month time constant, which we have determined is nominally optimum for most atomic clocks. The x$_{ij}$ denotes the measured time-differences between all the clocks for the current measurement cycle. After the invention of the Dual-Mixer Time Difference, the DMTD measurement noise was so low we went to much shorter measurement intervals for tau. Judah Levine then servoed a rubidium atomic clock to the software clock—giving us a real-time continuous output.

2) $$x_j(t_o + \tau) = \sum_{i=1}^{N} w_i \left[\hat{x}_i\,(t_o + \tau) - x_{ij}(t_o + \tau)\right]$$

Equation 3 then calculates the measured frequency for the jth clock over the last tau interval. This calculation is done for all the clocks, as the jth clock is representative of all the clocks. This is not the optimum estimate of the jth's fractional frequency, because cesium-beam atomic clocks are well modeled by white-noise frequency modulation (FM). On a σ$_y$(τ) diagram for a cesium-beam atomic clock, the knowledge of its true frequency improves as τ$^{-1/2}$ until it reaches what we call the flicker-floor. Flicker-noise FM causes σ$_y$(τ) to be proportional to τ^0. Then in the long-term the best noise model becomes random-walk FM. There σ$_y$(τ) is proportional to τ$^{+1/2}$. The AT1 algorithm accommodates these models.

3) $$y_j(t_o + \tau) = \frac{x_j(t_o + \tau) + x_j(t_o)}{\tau}$$

Then in Equation 4, we have shown that a digital exponential filter is optimum in this case and an optimum value of m$_j$ needs to be determined for each clock from a σ$_y$(τ) diagram of its performance. Fortunately, these diagrams only change slowly over time. So, Equation 4 then gives us the optimum estimate of the jth clock's fractional frequency offset. We apply this equation for all the clocks. Fortunately, this digital exponential filter is near optimum for flicker-noise FM as well. This is discussed in Chapter 9 of NBS Monograph 140, and in the paper, *A Study of the NBS Time Scale Algorithm: https://tf.nist.gov/general/pdf/816.pdf* .

4) $$\hat{y}_j\,(t_o + \tau) = \frac{y_j(t_o + \tau) + m_j\hat{y}_j(t_o)}{m_j + 1}$$

If on a σ$_y$(τ) diagram for a cesium-beam atomic clock, τ$_{min}$ is where the white-noise FM intersects the flicker-floor. Or, τ$_{min}$ is where the random-walk FM begins to be the dominate noise. Then a good estimate in the flicker case is m$_j$ = τ$_{min}$/τ. In the case of the best model for the cesium-beam atomic clock is white-noise FM and Randon-walk FM, then we may write equation 4b, which has been shown to be optimum.

4b)
$$m_j = \frac{1}{2}\left\{-1 + \left(\frac{1}{3} + \frac{4\tau_{min}^2}{3\tau^2}\right)^{1/2}\right\}$$

Equation 5 uses the calculations from equations 1 and 2 in the AT1 algorithm to estimate the errors for each clock. This is done by taking the difference between the optimum predicted value over the last measurement interval, τ, and the ensembles optimum estimate of the offset time of each clock. The last term in Equation 5 removes the bias of a clock looking at itself to the degree that it is a member of the ensemble and under the assumption of a normal distribution of errors.

5)
$$|\epsilon_i(\tau)| = \left|\tilde{x}_i(t_o + \tau) - x_i(t_o + \tau)\right| + \frac{0.8\langle\epsilon_e^2(\tau)\rangle}{\langle\epsilon_i^2(\tau)\rangle^{1/2}}$$

If the error is not an outlier, then it is squared and exponentially averaged into all previous values for that clock. The angle brackets, < >, denote a digital-exponential filtered estimate with a time constant of month. The solution from Equation 5 is the input into this ongoing digital-filtered estimate for each clock.

For equations 6 and 7, Jim Barnes showed me how to use **Lagrange's method of undetermined multipliers** to estimate the optimum weights to begin with. Once the AT1 atomic-clock ensemble algorithm was stated back in 1968, new clocks could be easily entered by calibrating their rates beforehand and entering them with a low weighting factor, so they would not pull the time of the ensemble off. The algorithm then finds the new clocks offset time, and exponentially learns its proper weight in the time-scale. The weights for all the clocks are updated at each measurement cycle, i = 1 to N.

6)
$$w_i = \frac{\langle\epsilon_e^2(\tau)\rangle}{\langle\epsilon_i^2(\tau)\rangle}$$

7)
$$where \langle\epsilon_e^2(\tau)\rangle = \left[\sum_{i=1}^{N}\frac{1}{\langle\epsilon_i^2(\tau)\rangle}\right]^{-1}$$

Like parallel resisters, the error of the ensemble will always be less than the error of the best clock. As new and better atomic clocks are added, the AT1 algorithm always keeps ahead. The frequency stability of AT1 is continually optimized.

Now, the big challenge is to include the outstanding performance of the new optical-frequency atomic clocks. Accuracies of 10^{-18} have been achieved. Running them as clocks is a real challenge, and some significant progress has been achieved there as well.

We wrote a paper back in 1975 on a way to achieve accuracy as well as frequency stability with intermittent frequency calibrations: https://tf.nist.gov/general/pdf/69.pdf . To test the theory, I made some significant improvements in the NBS-4 cesium-beam primary standard. I was able to improve the flicker floor by about a factor of ten. We got some nice results—consistent with the theory.

Recently, some of the scientists at NIST have gotten some very interesting results using some of the new optical frequency standards. They call it AT1'. There are several laboratories around the world who have developed optical frequency standards. I am pleased that that Patrizia Tavella, Director of Time Services at the BIPM, and Dr. Elizabeth Donley, Division Chief of the Time and Frequency Division in Boulder are studying this problem. From my perspective, it has a very bright and fruitful future.